Developing the Concept of the Linear Model

(Part One)

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EALR: 1.5 understand and apply concepts and procedures from algebraic sense relations and representations, and operations

Grade Level: 9-14

When teaching, I often reflect on how I was taught. Since I was rather successful in mathematics, I caught myself assuming that the way I was taught was pretty good. I have since discovered that most people don't learn like I do or even like anybody else does. Everyone has their own unique preconceived notions, misconceptions, understandings, experiences, preferred learning modalities, etc. making each person unique. Therefore, when presented with a classroom of fresh individuals eager (or not so eager) to learn, there needs to be a variety of approaches and experiences presented so the students can construct their own understandings. The teacher needs to build on students' understandings, past experiences, and address their modalities of learning. Bottom line is, there is no one right way to teach any material.

There are a variety of approaches to teaching the linear model. Many are very algorithmic and piece meal. For example, an algebra one class might spend a great deal of time and energy on developing the concept of slope. Then the direct variation model might be used to motivate the linear model followed by the slope-intercept model. Broken into pieces like this helps students manage the tasks and learn how the algorithms work but in this piece meal approach, often the overarching goal of the whole process is mysterious and unaddressed. Also, to clarify the basic algorithm often the context is stripped away. For some students this works. However many are left wondering what is going on and they often ask, "Where am I going to use this?" In an effort to give an answer, there needs to be context. Maybe not right away but in the end context needs to be a significant portion of the lessons, not an after thought. It can't be the problems at the end of a problem set that are left to the better students.

Other issues frustrate students. For example, $(y_2-y_1)/(x_2-x_1)$ is a nightmare for students. Not much clearer is rise/run or $\Delta x/\Delta y$. And why is $y=mx+b$ so important that teachers try to pound it into their students' heads. Teachers threaten with, "You'll never be able to do mathematics/pass high school/pass the WASL unless you can do this!" as if that will help students learn and understand. Some students learn the algorithm with no idea why they are doing it. (Some get very good at it, probably to please the teacher!) Some struggle through and pass the section. Others fail. But worst of all, few understand why they even were asked to do it.

After years of teaching the linear model and trying different approaches, I started having little 'Aha's!' Some of the 'Aha's' are so obvious that I'm embarrassed that I didn't realize them earlier. Others weren't so obvious. Some of what I've learned now appears in textbooks and may have already seen some the following material. So let's get started.

x	y
2	5
3	7
4	9

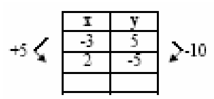
To scaffold on what the students have learned, let's start by looking at patterns. Students are quite good at patterns and I have learned to appreciate the elementary teachers and their hard work at building these skills. (It is unfortunate that often teachers don't scaffold on the skills taught students.) After all, the linear model is nothing more than a description of a pattern. Take for example the table (T-chart or whatever you want to call it). Ask the students what is going on in the x column. Without hesitation, they are quick to point out that it is going up by one. Ask them what the next two values would be. Again, they find this an easy task (thanks to their elementary teachers). Proceeding to the y column we get similarly quick responses. Now reflect on this question: Given point A(2, 5), point B(3, 7) and point C(4,9), find two more points that are on the line passing through these two points? We have just answered this question probably more easily than the many ways that could and probably would have been used.

Then ask the students to graph the three given points? Of course they need to be familiar with the Cartesian coordinate plane and plotting points on it. Sure there are the usual problems: x and y coordinates mixed up, plotting each coordinate as a separate point, etc. Students need to be helped through these common errors. There is a more important problem that may cause the students to develop misunderstandings. Consider how often, with students, not being very good at spacing the first unit or two on the graph in both the x and y direction, they get what appears to be widely varying slopes for the same three points. This is a good time to discuss this problem but it still doesn't go away. When using the table, however, they quickly generate a couple of extra points and the slopes tend to be much more accurate.

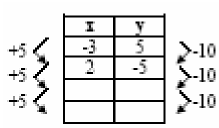
When the students have finished graphing their points ask them, "Where is the 2 that the y column changed by and the 1 that the x column changed by?" They then will respond with phrases like 'up' and 'over' (rise and run). It is much better for the students to discover the key ideas instead of being told them. Then it is just a matter of attaching a common name to them.

Consider this table. Any teacher having tried to teach slope using $(y_2-y_1)/(x_2-x_1)$ knows the nightmare the negative numbers cause for both the teacher and the students. In the table form, however, students are much more likely to correctly identify the pattern. They even begin using terms like, "the x column is changing by +5 and the y column is changing by -10." They aren't making the errors inherent with the negatives and the formula. They are able to focus on the pattern. Help them with notation by discussing the meaning of Δx and Δy and that the symbolism was created by lazy mathematicians unwilling to continually write change in x and change in y. After doing several of these tables, help students make the connection between Δx and x_2-x_1 and Δy and y_2-y_1. It is important to point out that the formalism is an off shoot of the concept instead of the formalism driving the concept.

x	y
-3	5
2	-5



Another point to be made is how to help the students notate their findings. For example, in the last table I would add the arrows and the notation shown to help them keep track of the patterns they describe.



It is useful to add the subsequent arrows helping them fill in the next boxes as well as emphasizing the change in x and the change in y ideas.

Also, continue to have the students graph the pattern and note where the +5 and the -10 appear. Notice that the slope concept is being developed but without giving it a name. Students will begin to want to come up with words to describe slope and then it is an easy step to name it and formalize the definition. Because everything up to now is based on what they are already familiar with i.e. patterning, students are able to understand this material in a lesson or two.

You may be asking where the context is or how the linear model concept is developed. Please hold off on context for just a bit more and let's develop the linear model. (This part I haven't seen in texts.) The following approach comes from trying to answer these questions, "Why is the x variable multiplied by the slope and why is slope rise over run?" or "Why is the change in x multiplied by the y and the change in y times the x?" The first questions are easier to answer after we get a handle on the second set of questions. The second questions arise when developing the general form of the equation of the line: $\Delta y(x) - \Delta x(y) = c$ or $ax + by = c$.

The tables become critical in helping students discover what was going on. Starting with a table like the one shown, ask the students to extend the table by at least two rows. Then ask, "What was the change in the x column (answer is add 2)?"

x	y
2	4
4	7
6	10

Have them create a 2x column by multiplying the x column by 2. Have them calculate the change in the 2x column (answer is 4). Then have the students create a 3x column and calculate its change (answer is 6). Ask them what would the x column be multiplied by so that it changed (grew) by ten (answer is 5)? By 20 (answer is 10)? Students quickly grasp this concept. Again, they are good at patterns.

3x	2x	x	y
6	4	2	4
12	8	4	7
18	12	6	10
24	16	8	13

Ask them, "What is the y column growing by?" and "How could they get both columns to grow by the same amount?" They are able to decide that there are several possible responses and that one that always works is multiplying Δy by x and Δx by y shown here where the change in x is 2 and the change in y is 3.

$\Delta y(x)$ or 3x	x	y	$\Delta x(y)$ or 2y
6	2	4	8
12	4	7	14
18	6	10	20
24	8	13	26

Then the critical question becomes, "What happens if the **difference** in the 3x column and the 2y column is calculated?" In this case they might respond with 2 or -2. In either case ask them to fill in the table with their answer as shown.

$\Delta y(x)$ or 3x	x	y	$\Delta x(y)$ or 2y	Difference
6	2	4	8	-2
12	4	7	14	-2
18	6	10	20	-2
24	8	13	26	-2

Since they may have calculated the difference by doing 3x-2y or 2y-3x, ask them to identify which they did by writing the expression that represents what they did below the word **difference**. In this example it would be 3x-2y. Then if 3x-2y is always -2 (or 2y-3x is always 2) then have

$\Delta y(x)$ or 3x	x	y	$\Delta x(y)$ or 2y	Difference 3x-2y
6	2	4	8	-2
12	4	7	14	-2
18	6	10	20	-2
24	8	13	26	-2

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them write the rule (answer is 3x-2y=-2). Of course in all good teaching it is important to recap what has just happened and to have them practice what **they** have **discovered**.

It is probably important at this point to mention that students now realize that x and y are standing for columns of numbers. This helps them develop a better concept of what a variable is. They begin to see that an equation is a description of the relationship between variables or columns of numbers. You may choose to talk about other values of the variables between the values shown on the table. You may also choose to discuss what happens if a student lists the points in reverse order. That is a common question from the students that want to know the one right way to do problems. Let them experiment and discover the result themselves.

Now for the other question, “Why is x multiplied by the slope?” Of course there are many ways to get to the slope-intercept form of the equation of the line. One is looking at context. A second is to solve for y in the general form above. Solving for y for many students is just playing magic with the symbols and is not enlightening. Let’s develop the slope-intercept form in the same fashion as we developed the general form; building on the ideas already developed. Considering our last table, ask “How could I get the x column to grow as fast as the y column?” Students familiar with making the changes in the rate of growth in the columns are able to do some calculations and determine that multiplying by 3 will get the x column to grow by 6 and then dividing by 2 will get it to grow by 3. They may do this in reverse order, divide by 2 to make the x column grow by one and then multiply by 3 to match the y. It makes no difference. The net result is that the x column is multiplied by 3/2 or 1.5. Have the students actually do the multiplication to get the (3/2)x column as shown. Ask them to compare the (3/2)x column and the y column. They notice that the y column is one bigger than the (3/2)x column. They then write that y=(3/2)x+1. Have them reflect on how they arrived at 3/2 and what might work in all cases. They usually realize that by dividing the x column by the change in x that the new column always changes by one. It is then a simple matter of multiplying by the change in y to get a column that matches the y column’s change. Now help the students notice that the resulting column will always vary by a constant amount from the y column. They are ready to write the general rule: $y=(\Delta y/\Delta x)x + b$. Armed with this information, students become powerful linear model makers.

(3/2)x	x	y
3	2	4
6	4	7
9	6	10
12	8	13

Before we go on, it is important to point out some other consequences/advantages that may not be anticipated. In working with the tables, there are many possible calculations that students and teachers may miscalculate. Without going into detail on the many possible mistakes, I urge you to experiment either by yourself or with your class on the possible mistakes. If you are doing it by yourself, imagine the errors that your students would make. Make them. Then see what the consequences of those mistakes cause. Because you or the students are developing multiple points on the line, graphing them, building models for them, creating columns for mx, $\Delta y(x)-\Delta x(y)$, etc, the students (and you) will automatically find virtually all of the errors. This method is self correcting. I can’t emphasize enough what a wonderful surprise this was for my students and me. I encourage you to try this with your students. You will be pleasantly surprised.

Part Two (in the next issue off Washington Mathematics Journal) will take us into that important part of adding context to the situations.

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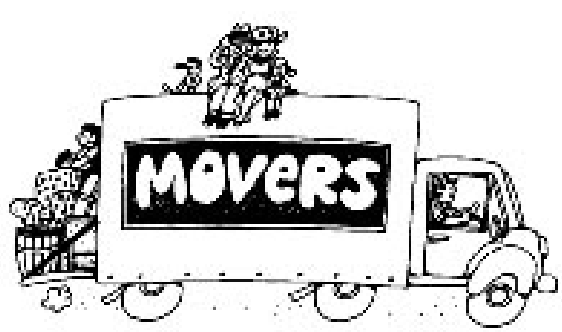

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